of a' from measured values of r', on the assumption that the value of the constant term is unity, it would be too large by the 0.00000 002 part. And this value substituted in the equation $n' = \sqrt{\frac{M}{a'^3}}$, would give n' too small by the 0.00000 003 part, or n' would be too small by $0''.03895$; or the error in the mean longitude of the sun would amount to nearly $4''$ in a century, a quantity which could not, in the present state of astronomy, be neglected. However, it is only fair to state that astronomers proceed in a way the reverse of this; that is they observe n' and thence deduce a', and in this case the term 0.00000 002 is without significance, since the logarithms of the radii vectores in the ephemerides are usually given to 7 decimals only.

A CASE OF SYMBOLIC VS. OPERATIVE EXPANSION.

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DENOTE by D^m the *symbolic* expansion of $\left(a_1\frac{d}{da_1}+a_2\frac{d}{da_2}+\ldots\right)^m$, the general term of which is $A_m a_1^r a_2^s \ldots \left(\frac{d}{da_1}\right)^r\left(\frac{d}{da_2}\right)^s\ldots$, where $A_m = \frac{m!}{r!\,s!\ldots}$, $r+s+\ldots = m$; and by $(D)^m$, the operation D or $\left(a_1\frac{d}{da_1}+a_2\frac{d}{da_2}+\ldots\right)$ repeated m times, whose general term is

$$A_m\left(\left(a_1\frac{d}{da_1}\right)\right)^r\left(\left(a_2\frac{d}{da_2}\right)\right)^s\ldots;$$

the *extra* parenthasis here, and in what follows, inclosing a symbol which combines *operatively*. Then $(D)^2$ differs from D^2 in the production of an extra term $a_i\frac{d}{da_i}$ by each portion $(D)a_i\frac{d}{da_i}$ $\left[\text{from}\left(a_i\frac{d}{da_i}\right)a_i\frac{d}{da_i}\right]$ of the complete operation $(D)\,D$ or $(D)^2$; so that the complete $(D)^2 = D^2 + D$. And in general, $(D)\,D^{m-1}$ differs from D^m by an extra term

$$(r+s+\ldots)A_{m-1}a_1^r a_2^s\ldots\left(\frac{d}{da_1}\right)^r\left(\frac{d}{da_2}\right)^s\ldots$$

derived from each portion

$$(D)\,A_{m-1}a_1^r a_2^s\ldots\left(\frac{d}{da_1}\right)^r\left(\frac{d}{da_2}\right)^s\ldots$$ of the complete operation $(D)\,D^{m-1}$; so that, since $r+s+\ldots=m-1$, $(D)D^{m-1}=D^m+(m-1)D^{m-1}$.

$\therefore\ D^m=[(D-m+1)]D^{m-1}=[(D-m+1)][(D-m+2)]\ldots[(D-1)]D$ or $(D)^{m'}$. (1)

Inversely, to find $(D)^m$ in terms of $D^m, D^{m-1}, .. D$, we have m equations linear in $(D)^m, (D)^{m-1}, .. (D)$ (obtained by assuming for m in (1) the values $1, 2, 3, .. m$) from which, if $(m-k)_i$ = the sum of the products of the natural numbers from 1 to $m-k$ inclusive taken i at a time,

$$(D)^m = \begin{vmatrix} D^m & -(m-1)_1 & (m-1)_2 & \cdots & \mp(m-1)_{m-1} \\ D^{m-1} & 1 & (m-2)_1 & \cdots & \pm(m-2)_{m-2} \\ D^{m-2} & 0 & 1 & \cdots & \mp(m-3)_{m-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ D & 0 & 0 & \cdots & 1 \end{vmatrix}$$

By another familiar process applied to the symbols D^m [$=(D)^{m\prime}$] and $(D)^m$ we get also

$$(D)^m = \Sigma \frac{\Delta^i 0^m}{i!} D^i.$$

Examples of (1).—1. To change the independent variables in D^m from $\alpha_1, \alpha_2, ..$ to $\theta_1, \theta_2, ..$, where $\alpha_1 = \varepsilon^{\theta_1}$, $\alpha_2 = \varepsilon^{\theta_2}, ..$; so that

$$\alpha_1 \frac{d}{d\alpha_1} = \frac{d}{d\theta_1},\ \alpha_2 \frac{d}{d\alpha_2} = \frac{d}{d\theta_2}, \ldots \qquad \left(\left(\alpha_1 \frac{d}{d\alpha_1} + \alpha_2 \frac{d}{d\alpha_2} + ..\right)\right)^{m\prime}$$

becomes by the transformation

$$\left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + ..\right)^{m\prime}. \quad \therefore \text{ by (1) } D^m = \left(\frac{d}{d\theta_1} + \frac{d}{d\theta_2} + ..\right)^{m\prime}.$$

For a special case see Todhunter's Dif. Cal. Art. 208.

2. By Euler's theorem concerning a homogeneous function $\varphi(\alpha_1, \alpha_2, ..)$, of n dimensions, $F((D))\varphi = F(n)\varphi$; $\therefore$ $(D)^{m\prime}\varphi = n^{m\prime}\varphi$; $\therefore$ by (1)

$$D^m \varphi = n^{m\prime}\varphi.$$

SOLUTION OF A PROBLEM.

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Problem—To show that $\cos^p\varphi \sin^q\varphi$ can be expanded into a series of *cosines* of multiples of φ when q is *even*, and into a series of *sines* of multiples of φ when q is *odd*.

First, suppose q even and $= 2n$, say. Then

$$\cos^p\varphi \sin^{2n}\varphi = \cos^p\varphi(\sin^2\varphi)^n = \cos^p\varphi(1-\cos^2\varphi)^n$$

$$= \cos^p\varphi - n\cos^{p+2}\varphi + \frac{n(n-1)}{2!}\cos^{p+4}\varphi - \&c.$$

Each term of this series can be expanded into a series of cosines of multiples of φ.